On the Special Properties of Graphic and Co-graphic Bondgraphs

by S. H. BIRKETT

School of Engineering, University of Guelph, Guelph, Ontario, Canada N1G 2W1

ABSTRACT: *The special combinatorial properties of graphic and co-graphic bondgraphs are analysed. A graphic bondgraph is one which has an associated graph, meaning that the two represent the same combinatorial information, expressed in the junctions of the bondgraph or the cycles of the graph. Dually a co-graphic bondgraph is one whose dual bondgraph has an associated graph. The combinatorial basis for constructing a systematic bondgraph model is explained, leading to the definitions of vertex and mesh bondgraphs and their duals. For each part of a vertex or mesh bondgraph one junction, called an auxilliary junction, and all its internal bonds may be eliminated. This procedure, called grounding a node in the bondgraph literature, is a purely combinatorial procedure and has no physical interpretation, nor does the choice of the auxilliary junction affect the meaning and interpretation of the effort and flow variables of the elements of a bondgraph model. Inverting this construction of a systematic bondgraph motivates the definition of nodal and mesh bondgraphs, with special properties parallel to those expressed in the vertex structure of a graph. It is shown how to construct a nodal (mesh) bondgraph from any graphic (resp. co-graphic) bondgraph and extend these by adding new external bonds called nodal (resp. mesh) bonds. This procedure may be applied to a bondgraph model to yield alternative formulation methods, pseudo-nodal (resp. pseudo-mesh) formulations, with precisely the same advantages as the nodal (resp. mesh) techniques of network analysis.*

I. Introduction

Combinatorial methods of physical system modelling combine physical information, i.e. how the components behave, with topological information, i.e. how the components are connected. This latter information is characterized by a collection of cycles and a dual collection of co-cycles. Each *cycle* or *co-cycle* is simply a set of components on which something analogous to Kirchhoff's laws are satisfied. The collection of all the cycles is an example of a mathematical construct known as a *matroid* (the co-cycle collection is then its dual matroid). The cycles and co-cycles may be represented as a pair of matrices, known as a *matrix representation* of the matroid. Equations are formulated by combining the topological and physical information.

Visual presentation of the combinatorial information is achieved in two distinct ways, as a *graph* or as a *bondgraph*, and this leads to two apparently distinct methods. A bondgraph presents just the cycles and co-cycles and contains no additional topological information. A graph presents more than the cycles and co-cycles, which must be extracted from its combinatorial structure. In particular, a graph also possesses *vertices* (*nodes*) which may be used to advantage even though they are neither an inherent nor essential aspect of the topology of a physical system. If the graph is planar then it also possesses meshes (faces) which are really the vertices of a dual graph. Kron (1) has even argued that graphs are illegitimate models of electrical networks, but, even though the nodes are not part of a physical system, their inclusion does not affect the matroid information. Nodal (and mesh) techniques are well-known, powerful analytical tools in network analysis and, in general, for any graph-theoretic model. It would appear then that since vertices are not available in a bondgraph, nodal techniques should not be possible with bondgraphs. In fact, it is quite simple to define pseudo-nodal variables for a bondgraph, and in this paper the mathematical basis for these is explained.

In (2–5), an independent theory is developed to analyse the combinatorial properties of bondgraphs. Specifically, a *combinatorial bondgraph* consists of junctions and non-directed bonds, like a junction structure but with no *TF* or *GY* components. The standard meanings of the junctions define a collection of cycles from which the remaining cycles can be generated, thus the diagram is a direct visual presentation of the topological information of the physical system. A summary of this theory is included in the Appendix, and the terminology and notation of the original work is freely used here. It should be pointed out that in this theoretical context the junctions are labelled with the letters "s" and "p" rather than "1" and "0", because the junction symbols are an integral part of the algebraic notation used in the theory and numerical symbols would be confusing.

In this paper, the special properties of graphic and co-graphic bondgraphs are discussed. A *graphic* bondgraph B is one which has an associated graph $G(B)$, which means that B and G represent precisely the same matroid, called the *cycle matroid* of either B or G, denoted by $M(B)$ and $M(G)$, respectively. A *co-graphic* bondgraph is one for which the dual bondgraph B^* has an associated graph. In applications, a bondgraph is often both graphic and co-graphic and thus has an associated planar graph. For a bondgraph which is only co-graphic all of the work of this paper may be applied in dual form to the (graphic) dual bondgraph. Hence, this paper describes techniques which may be applied to a bondgraph which is at least graphic or co-graphic, but possibly both.

Section II begins by developing, in the general context of combinatorial bondgraphs, the systematic method of obtaining bondgraphs from an associated graph. Such bondgraphs are called *systematic* bondgraphs and are constructed directly from a diagram representing the graph, which, in general, need not be a planar representation. If the graph, G, is planar it has a dual graph, G^*, this situation corresponding to a bondgraph which is both graphic and co-graphic, and the dual of the method uses G^* to construct another bondgraph, different from, but equivalent to, the bondgraph obtained from G. A systematic bondgraph contains redundant information, and it is possible, for each part of a systematic bondgraph, to delete one junction and the incident internal bonds without affecting the cycle matroid $M(B)$ represented by the bondgraph. This procedure is called "grounding a node" in the bondgraph literature, which refers to the subsequent simplification of a systematic bondgraph model.

Section III shows how a vertex-like structure may be extracted from a graphic bondgraph, without the construction of any associated graph. By a suitable transformation of the bondgraph it is possible to select a collection of bonds which provide a base for the cycle matroid of the bondgraph, with the property that all the fundamental cycles have a maximum of two base elements. This is precisely the same requirement as for a Lagrangian tree of a graph, and this base may be used to define pseudo-nodal variables for a graphic bondgraph model, used in the same way as the nodal variables of network analysis or graph-theoretic modelling. These results demonstrate the existence of pseudo-nodal variables for any graphic bondgraph, but, in fact, it is not necessary to construct a nodal bondgraph in order to define and use them. For bondgraphs which are co-graphic the dual of this provides methods which are analogous to the mesh methods of network analysis or graph theoretic modelling. However, in the case of a bondgraph model, the dual technique is conceptually the same as the primal.

II. Systematic Bondgraphs

Let G be a graph. Replace each vertex of G by a p-junction, divide each edge of G in the centre by an s-junction, and add an external bond to each of these new s-junctions, labelled, for convenience, the same as the corresponding edge of G. The bondgraph constructed in this way is called the *vertex bondgraph* of G and will be denoted by $B_+(G)$. By interchanging the reference to p- and s-junctions in the definition of $B_+(G)$ the dual bondgraph, denoted by $B_+^*(G)$, is constructed. This dual bondgraph is called the *dual vertex bondgraph* of G.

This construction of $B_+(G)$ is motivated by the systematic method of obtaining a bondgraph model of a physical system [see Thoma (**6**)]. The construction of a vertex bondgraph exactly corresponds to the construction of a systematic bondgraph model from a circuit diagram of the physical system. In the case of the mechanical domain it is customary to interchange the junctions and thus systematic mechanical models use the dual vertex bondgraph.

Theorem I

The vertex bondgraph $B_+(G)$ is associated with the graph G.

Proof: For each cycle of G add the elementary junction vectors of $B_+(G)$ corresponding to the vertices of the cycle. All the internal bonds will be eliminated since the p-junctions have only internal bonds. In this way, a one-to-one correpondence is established between cycles of G and the vectors in the cycle space of $B_+(G)$. ∎

Theorem II

Let B_+ be the vertex bondgraph of a graph G. Then the rank and co-rank of B are given by

$$\rho(B_+) = p-s+e_s-\operatorname{sep}(G) = p-\operatorname{sep}(G)$$

$$\rho^*(B_+) = s-p+e_p-\operatorname{sep}(G) = s-\operatorname{sep}(G)$$

where sep (G) denotes the number of parts of G. (The other symbols are defined in the Appendix.)

Proof: Since G and B_+ are associated they must have the same rank, which, from elementary graph theory, is $v(G) - \text{sep}(G)$ (this is the number of edges in a forest of G), where $v(G)$ is the number of vertices of G. By construction this is $p - \text{sep}(G) = p - s + e_s - \text{sep}(G)$, since $s = e_s$ for a vertex bondgraph. The dual vertex bondgraph of G provides the second result. ∎

Corollary 1

Any pseudo-base colouring (causal assignment) of a vertex bondgraph must have a red–white loop (causal loop).

Proof: If a pseudo-base colouring of a vertex bondgraph has no red–white loop then it is a base colouring and so the bondgraph has rank $p - s + e_s = p$ [see (**5**)], contrary to the result of the theorem. ∎

Theorem III

Let β be a pseudo-base set obtained from a pseudo-base colouring of a vertex bondgraph of a graph with p vertices. Then β contains p elements.

Proof: For any pseudo-base colouring of a vertex bondgraph there must be precisely one red (internal) bond on each p-junction and hence one such bond on each s-junction. This will provide precisely p white external bonds on s-junctions, which proves the result. ∎

These results demonstrate that a vertex bondgraph is not appropriate for assigning causality. By examining an augmented graph for B_+ it becomes clear why this is so, because the augmented graph possesses an extra vertex on which no external edges are incident. An *augmented* graph of a bondgraph is a graph which also includes edges corresponding to the internal bonds of B.

Theorem IV

Let B_+ be the vertex bondgraph obtained from a connected graph G with p vertices. An augmented graph for B_+ consists of G together with internal edges incident on a common extra vertex g. Let p_a be one of the p-junctions of B_+, corresponding to the vertex a of G. Then each internal bond on p_a corresponds to an internal edge of G incident on g and a. If G is not connected then these results apply separately to each part of G.

Proof: Suppose, first, that G is a connected graph. By construction, every p-junction of B_+ will be an internal junction and every internal bond will be incident on some p-junction. Since B_+ and G are associated, any external graph for B_+ will require p vertices, the same as the graph G. By appending one external bond to each of the p-junctions of B_+ a bondgraph B'_+ is constructed with no internal junctions and precisely one external bond on each of its junctions. Furthermore, a graph associated with B'_+ will be topologically identical to the augmented graph of B_+, each collection of parallel internal edges arising from a given p-junction being parallel to the appended external edge on that junction. By colouring every one of the external bonds of B'_+ red a base colouring of B'_+ is obtained and so $\rho(B'_+) = p - s + e_s = p$. Thus, a graph associated with B'_+ will require $p+1$ vertices and so will an augmented graph for B_+. Now, by considering the p-space of B'_+, a co-cycle can be found consisting of all the appended external bonds:

- take the combination of all the dual elementary junction vectors on the p-junctions of B'_+;
- this combination includes all the appended external bonds and every internal bond;
- eliminate the internal bonds by adding s^*-type elementary junction vectors consisting of the two internal bonds on each s-junction;
- the remaining co-cycle consists of every appended external bond and no internal bond.

From this it is concluded that a graph associated with B'_+ consists of G, together with an extra vertex on which are incident the p edges corresponding to each of the appended external bonds.

By considering the cycle space of B'_+ it is seen that each external bond on an s-junction forms a cycle with the two appended external bonds on the neighbouring p-junctions. Thus, no pair of edges corresponding to appended external bonds can be parallel in an associated graph of B'_+ and so each of these must be incident on a different vertex.

Finally, since this graph is topologically equivalent to an augmented graph of B_+, this augmented graph must consist of G, together with mutually parallel collections of internal edges incident on a common extra vertex, one collection incident on each of the p vertices of G.

If G is not connected, note that the connectivity of the underlying graph of B_+ is the same as that of G. In this case, the proof above applies separately to each of the parts of G, each part of G having a separate extra vertex. ■

The extra vertex present in each part of an augmented graph for a vertex bondgraph will be called an *auxilliary vertex*. The next theorem provides the fundamental bondgraph technique called "grounding a node" in the bondgraph literature [see Thoma (**6**)]. It also explains the significance of this procedure in terms of the associated graph.

Theorem V

Let B_+ be the vertex bondgraph of a graph G. If, in each part of B_+, one p-junction and all the incident (internal) bonds on that p-junction are deleted, the resulting bondgraph is equivalent to B_+, in particular it is also associated with G. Let the deleted p-junction, p_a, be the junction corresponding to vertex a of the graph G. In terms of an augmented graph for B_+ the procedure of deleting p_a and its internal bonds corresponds to contracting the corresponding internal edges of the augmented graph: the internal edges incident on a are deleted and the auxilliary vertex is collapsed onto the vertex a.

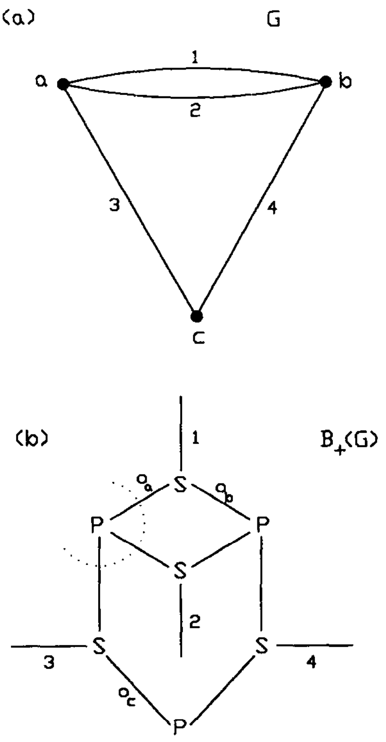
Proof: By considering the augmented graph for B_+ constructed in Theorem IV it is clear that deleting p_a from B_+ corresponds to collapsing the auxilliary vertex onto vertex a, since the internal edges are removed from any cycles which contain elementary junction vectors on neighbouring s-junctions to p_a. This procedure may be done once for each of the auxilliary vertices of B_+, i.e. once for each part of G. ■

Corollary 2

Let B_+ be the vertex bondgraph of the graph G. In each part of B_+, one p-junction and all the incident (internal) bonds on that p-junction may be deleted without altering the cycle and co-cycle matroids of B_+.

Let B_+ be the vertex bondgraph of the graph G. Deleting p_a, the p-junction corresponding to vertex a of G, with its incident bonds, is called *grounding* the p-junction p_a or the vertex a. The deleted junction is called an *auxilliary junction.* The resulting bondgraph is denoted by B_a and is called a *reduced vertex bondgraph* for G, as is any equivalent bondgraph obtained from B_a by contraction of internal bonds joining similarly labelled junctions. The procedure of grounding a vertex is called *reduction* or *simplification.*

Example 1. Figure 1 illustrates the construction of a vertex bondgraph B_+, for a graph, G, and its subsequent reduction by grounding a vertex. For simplicity, the



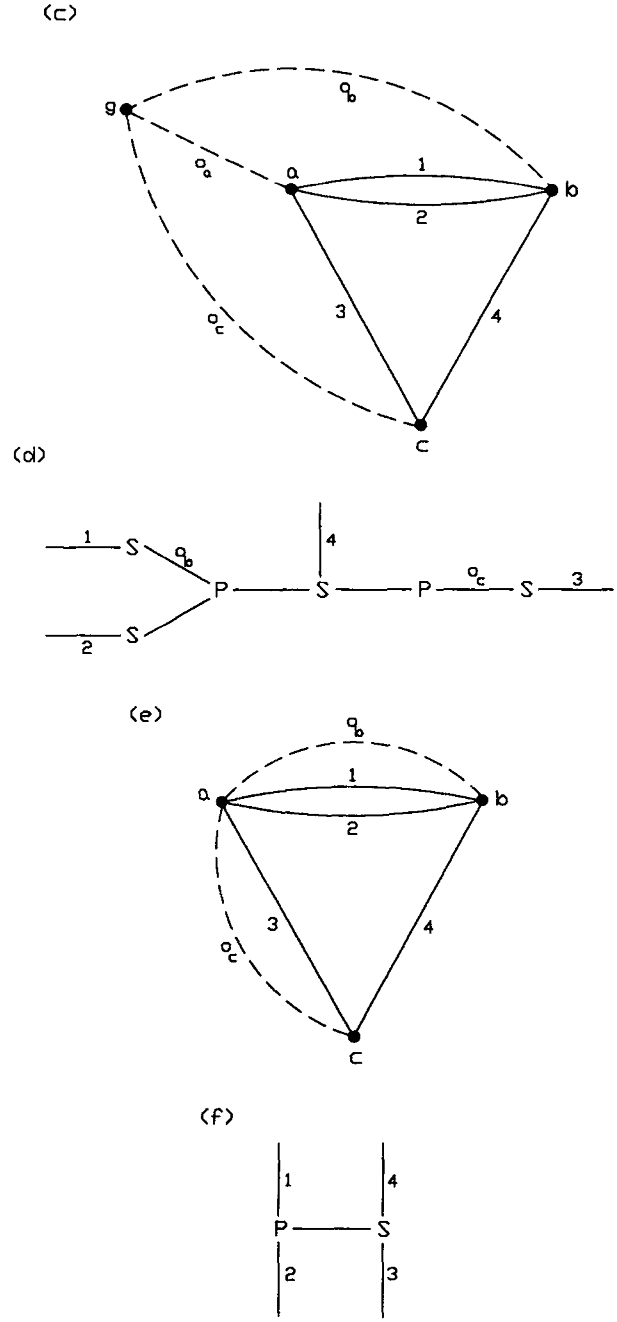
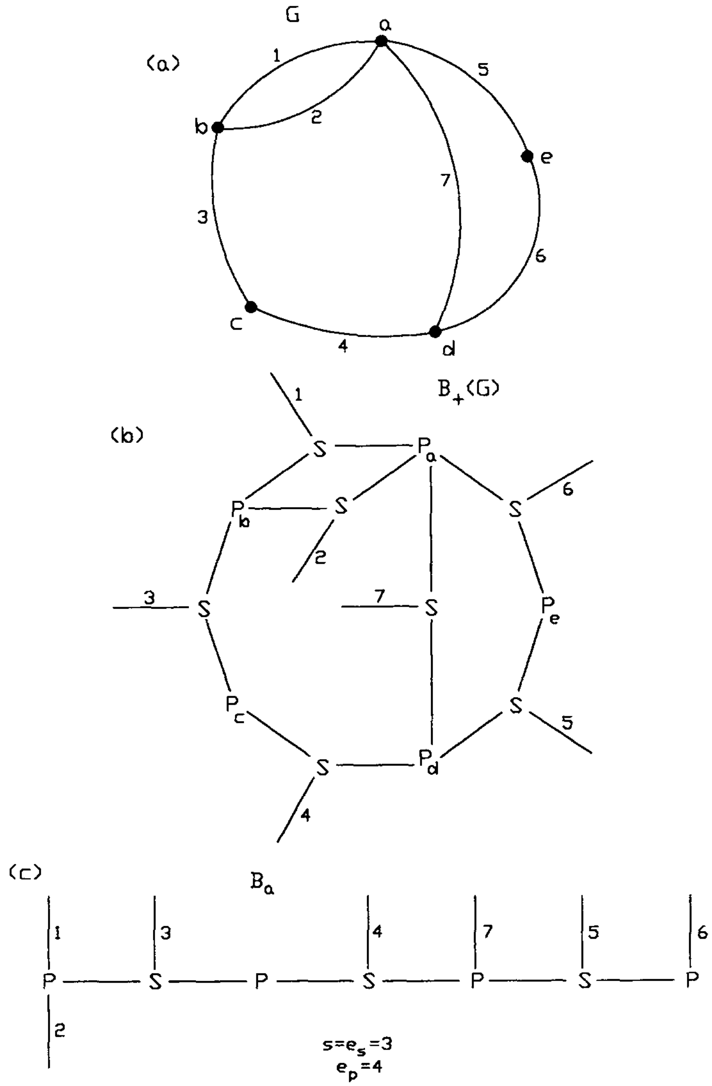


FIG. 1. Construction and simplification of a vertex bondgraph: (a) graph G; (b) vertex bondgraph B_+; (c) augmented graph for B_+; (d) B_a: reduced vertex bondgraph with vertex a grounded; (e) augmented graph for B_a; (f) simplified (proper) bondgraph equivalent to B_a and B_+.

augmented graph for B_+ is shown here with only one edge representing each collection of parallel edges: edge o_a, for instance, represents the three parallel internal edges associated with the p-junction at vertex a of G. The reduced vertex bondgraph B_a is obtained by grounding the vertex a. The procedure of grounding this vertex corresponds to collapsing the auxilliary vertex, g, onto vertex a in the augmented graph, as shown in Fig. 1(e). A simplified proper bondgraph equivalent to both B_+ and B_a is given in Fig. 1(f).

Example 2. In this example a vertex bondgraph, B_+, is constructed in Fig. 2 and subsequently simplified to reduced vertex bondgraphs, corresponding to grounding each one of the five vertices in turn. The reduced vertex bondgraphs are not



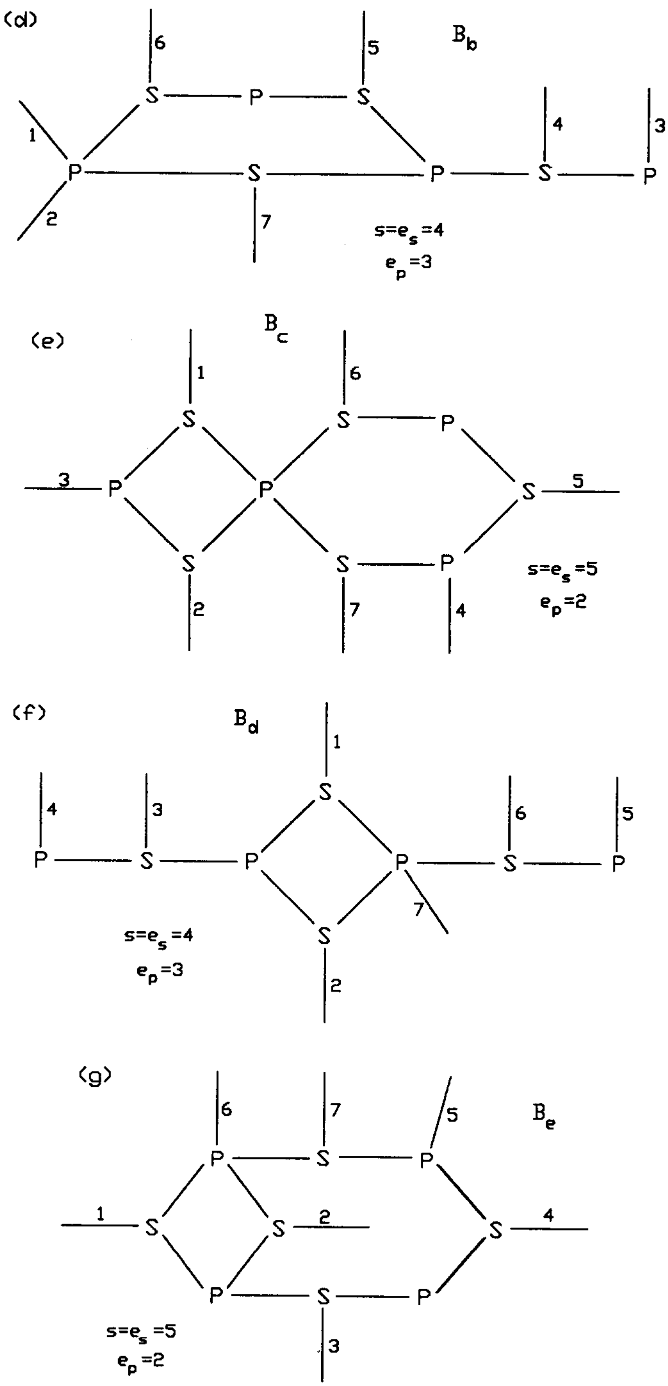


FIG. 2. Construction and simplification of a vertex bondgraph: (a) graph G; (b) vertex bondgraph B_+; (c)–(g) five reduced vertex bondgraphs for B_+.

simplified to their proper contractions. It would be simple to do so but, for purposes of comparison, the reduced vertex bondgraphs are simplified only to the point where they have four p-junctions and each s-junction has one incident external bond and two incident internal bonds. The reduced vertex bondgraphs have the following properties:

- B_e and B_c, each with two loops, have $s = e_s = 5$ and $e_p = 2$;
- B_b and B_d, each with one loop, have $s = e_s = 4$ and $e_p = 3$;
- B_a with no loop has $s = e_s = 3$ and $e_p = 4$.

A careful choice of which vertex to ground will produce a simplified bondgraph with the least number of loops.

Let G be a planar graph with dual graph G^*. The *face bondgraph* of G, denoted by $B_-(G)$, is the dual vertex bondgraph of G^*. An s-junction corresponding to one of the faces of G is *grounded* if it is deleted with its incident bonds. The deleted junction is called an *auxilliary junction*, and the resulting bondgraph is called a *reduced face bondgraph* for G, as is any equivalent bondgraph obtained from it by contraction of internal bonds joining similarly labelled junctions. The procedure of grounding a face is called *reduction* or *simplification* of the face bondgraph.

A *systematic bondgraph* for a graph G is the vertex bondgraph of G, the dual vertex bondgraph of G or, if G is planar, the face bondgraph of G or its dual.

The existence of a dual graph is equivalent to the case of a graphic and cographic matroid. The face bondgraph may be constructed directly from a planar representation of a graph in the same manner as the construction of a geometric dual. The faces of a graph will be labelled by their boundary edges. For instance, face 124 of a graph denotes the face with boundary consisting of edges 1, 2 and 4 in the given planar representation. The reduced face bondgraph of B_-, obtained by grounding this face, would be denoted by B_{124}.

Theorem VI

Let B_- be the face bondgraph of a planar graph G. Then B_-, or any reduced face bondgraph, is equivalent to the vertex bondgraph of G, $B_+(G)$. In particular, the vertex and face bondgraphs, or any reduced versions of them, are all associated with G.

Proof: The results of the theorem follow from the definitions and the results of (**4**). ■

The bondgraph constructed from a planar graph, G, using the cut and paste method of (**2**) is a special case of a reduced face bondgraph, the one in which the unbounded face of G is grounded.

Example 3. In Fig. 3, the vertex and face bondgraphs of a planar graph are constructed. All the reduced versions of these are also shown, demonstrating that, in general, reduced vertex and reduced face bondgraphs of a graph are topologically distinct, although, in some cases, these may coincide.

The reduction of a vertex or face bondgraph by grounding an auxilliary vertex is a purely combinatorial procedure. When applied to a bondgraph model of a physical system there is no physical interpretation of this procedure. Neither the meaning nor the interpretation of the variables of the bondgraph elements (these correspond to the external bonds) are affected by "grounding a node" nor by the choice of auxilliary vertex. The procedure simply corresponds to the elimination of redundant information relating the internal bonds. It is quite incorrect to regard "a grounded vertex" as a physical ground or datum with respect to which the bondgraph variables are measured. The bondgraph variables of a vertex bondgraph or any one of its reduced versions have precisely the same respective values and only the internal bond structure is different in these bondgraphs.

III. Lagrangian Bases for Bondgraphs

Let B be a bondgraph in which every internal bond joins junctions of an opposite type. B is called a *nodal bondgraph* if precisely one external bond and two internal bonds are incident on each s-junction. Dually, B is a *mesh bondgraph* if precisely one external bond and two internal bonds are incident on each p-junction.

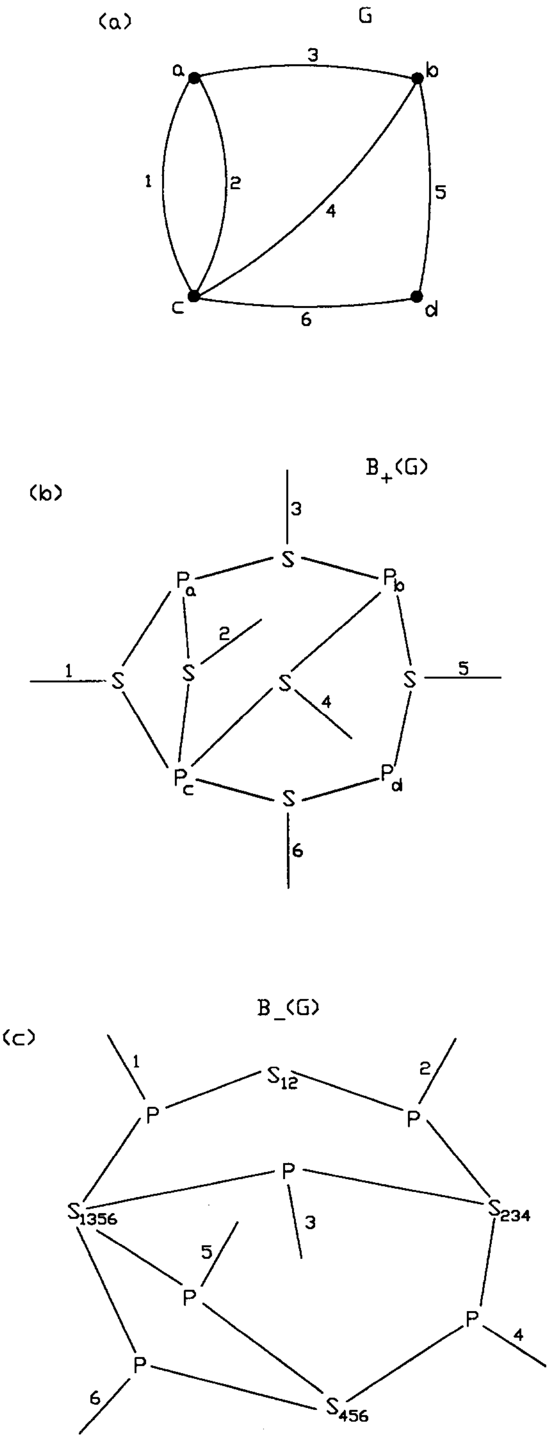
A vertex bondgraph of a graph is a nodal bondgraph, as is a reduced vertex bondgraph obtained directly from grounding a p-junction. Similarly, a face bondgraph of a planar graph is a mesh bondgraph, as is a reduced face bondgraph obtained directly from grounding an s-junction. The definitions are motivated by consideration of these constructions. However there is also a very significant algebraic motivation.

In this section, the procedure of Section II is inverted and, beginning with a bondgraph, a nodal or mesh bondgraph equivalent to it is constructed. First, conditions are determined which guarantee the existence of equivalent nodal bondgraphs and equivalent mesh bondgraphs.

Since bondgraphs are, in essence, pictures of binary matroids, it is natural that many of the results for bondgraphs will reflect general matroid results. Before discussing the construction of nodal and mesh bondgraphs, a useful matroid concept is introduced. Recall that cycles and co-cycles are called circuits and co-circuits in a general matroid. A set of circuits of a matroid M is called a *generating set of circuits* if every circuit in M may be obtained from it by symmetric differences (see the Appendix for this definition). A *minimal generating set of circuits* is a generating set such that no subset is a generating set of circuits. A *2-generating set of circuits of M* is a minimal generating set of circuits such that an element belongs to, at most, two of the circuits. Generating, minimal generating and 2-generating sets of co-circuits of M are defined similarly. An important matroid result states that a binary matroid is graphic if and only if it has a 2-generating set of co-circuits (7). Dually a binary matroid is co-graphic if and only if it has a 2-generating set of circuits. This theorem is used to prove a general result for graphic and co-graphic bondgraphs.

Theorem VII

A bondgraph is graphic if and only if there exists an equivalent nodal bondgraph. Dually, a bondgraph is co-graphic if and only if there exists an equivalent mesh bondgraph.


(a)
G
a
b
c
d
1
2
3
4
5
6
(b)
B+(G)
Pa
Pb
Pc
Pd
S
1
2
3
4
5
6
(c)
B_(G)
S12
S1356
S234
S456
P
1
2
3
4
5
6

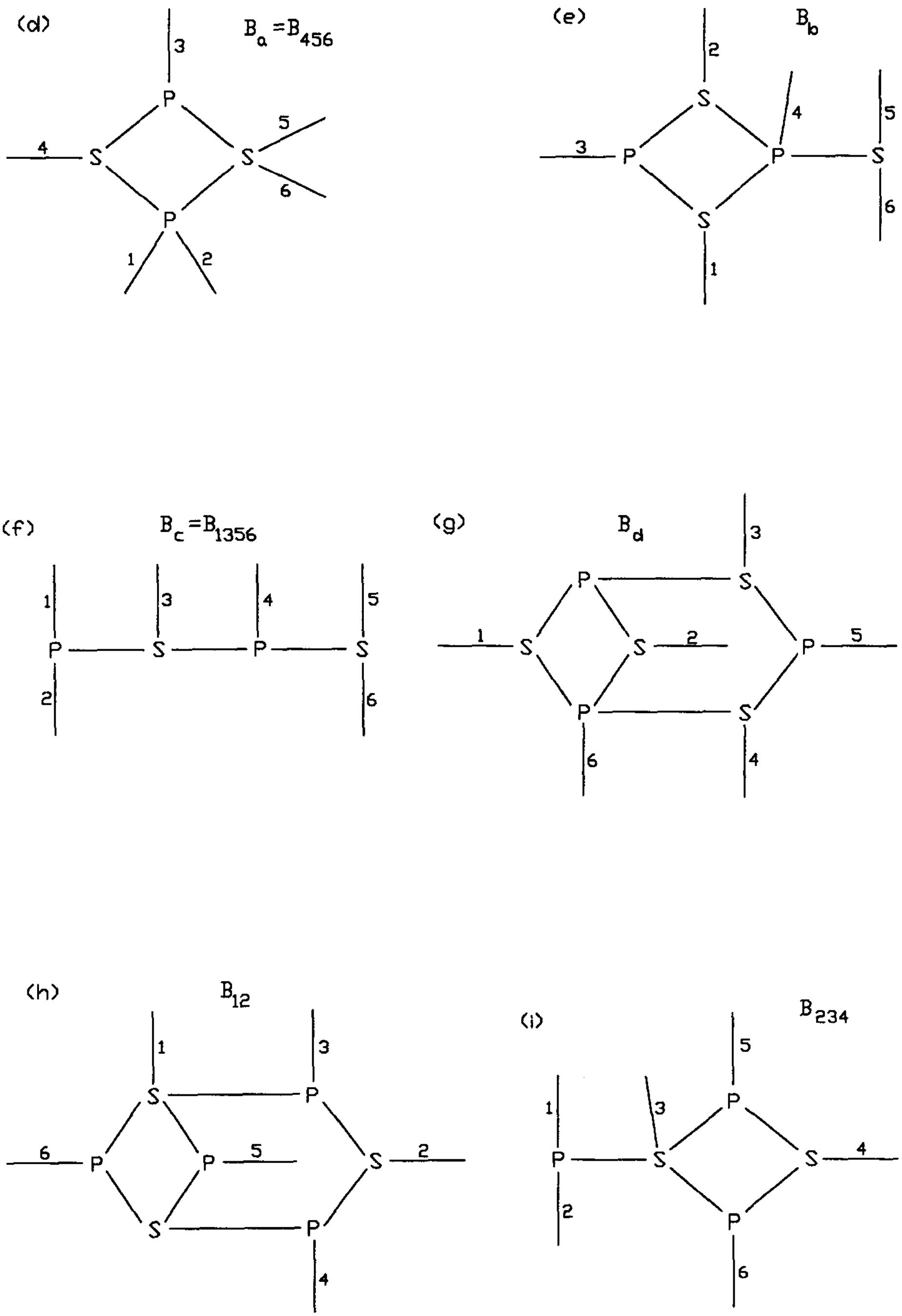


FIG. 3. Vertex and face bondgraphs of a planar graph and reduced versions: (a) graph G; (b) vertex bondgraph $B_+(G)$; (c) face bondgraph $B_-(G)$; (d)–(i) reduced vertex and reduced face bondgraphs.

Proof: Let M be the cycle matroid of a nodal bondgraph equivalent to B. By the definition of equivalent bondgraphs $M \cong M(B)$, the cycle matroid of B. A minimal generating set of co-circuits of M can be obtained from the dual elementary junction vectors of the nodal bondgraph. Furthermore, bonds attached to a p-junction will appear in only one co-circuit of the generating set and bonds attached to an s-junction will appear in two different co-circuits of the generating set. Thus, this generating set is a 2-generating set of co-circuits of M. Since the matroids of a bondgraph are binary matroids, the matroid M is graphic and hence also the cycle matroid of B is graphic, from the general matroid result stated above. The converse is true, for a graphic bondgraph, B has an associated graph and the vertex bondgraph of this graph is a nodal bondgraph equivalent to B.

The dual result is proved by using the s-junctions of the equivalent mesh bondgraph to construct a 2-generating set of circuits of M, which shows that B must be co-graphic. ■

Example 4. Figure 4 shows a bondgraph, B, and an equivalent nodal bondgraph. The nodal bondgraph is easily constructed from B by simply expanding the s-junctions with more than three incident bonds and creating a new p-junction. It is possible to do this since each of the s-junctions has two incident internal bonds.

The construction of an equivalent mesh bondgraph to B, shown in Fig. 5, is more involved since one of the p-junctions has three incident internal bonds. To remove one of these three internal bonds requires the p-junction to be coupled with one of the neighbouring s-junctions, thus producing an equivalent bondgraph with a loop ["diamond equality" (**2**)]. This coupling procedure, using the s-junction with incident bonds 1 and 2, is shown in Fig. 5(b), which shows the extracted sub-bondgraph and its coupled version. In Fig. 5(c), the coupled version of the extracted

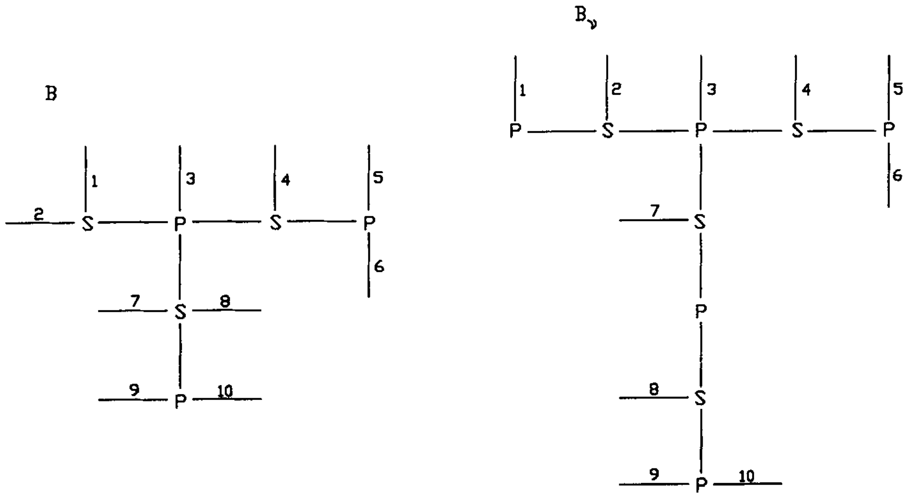


FIG. 4. Bondgraph and equivalent nodal bondgraph.

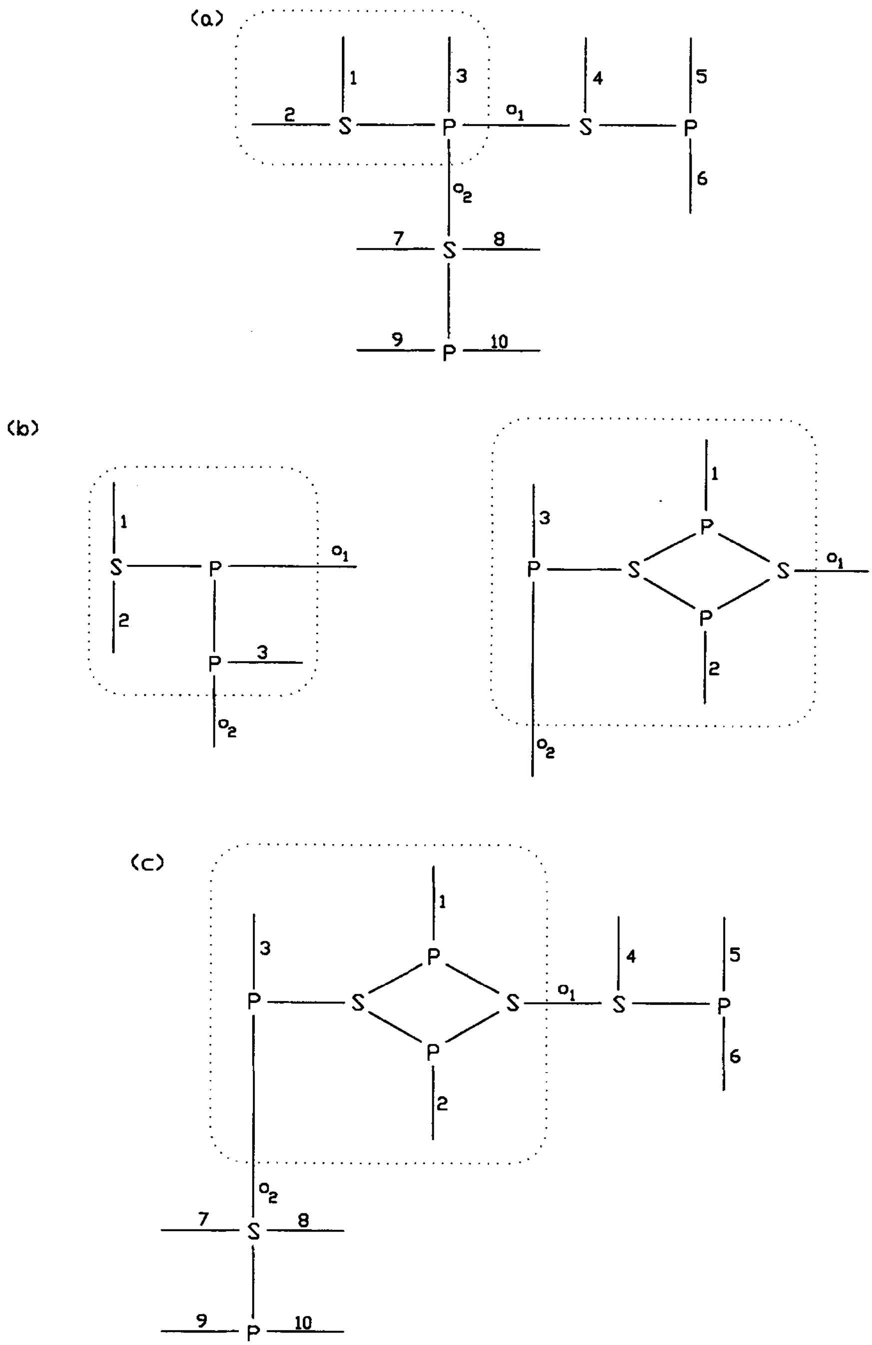


FIG. 5. Bondgraph and construction of equivalent mesh bondgraph: (a) bondgraph B; (b) extracted sub-bondgraph and coupled version; (c) sub-bondgraph re-connected into B; (d) final equivalent mesh bondgraph. *Continued overleaf.*

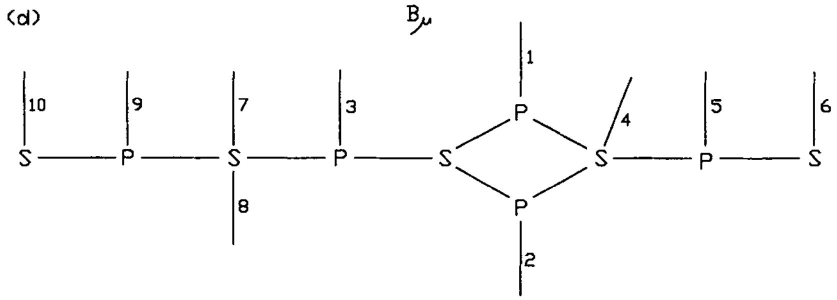


FIG. 5. *Continued.*

sub-bondgraph is re-connected into the correct place in B. Finally, after a simple expansion on the other two p-junctions and contraction of the internal bond labelled o_1 which now joins two s-junctions, the mesh bondgraph of Fig. 5(d) is constructed.

Let B be a graphic bondgraph. B_v denotes any nodal bondgraph equivalent to B. The *extended nodal bondgraph* for B_v, denoted by B'_v, is constructed by attaching one extra external bond to each p-junction of B_v. These extra bonds are called *nodal bonds* and are labelled $v_1, v_2, \ldots, v_{p(B_v)}$. Dually, for a co-graphic bondgraph, B, an equivalent mesh bondgraph will be denoted by B_μ. The *extended mesh bondgraph* for B_μ, denoted by B'_μ, is constructed by attaching one extra external bond to each s-junction of B_μ. These extra bonds, labelled $\mu_1, \mu_2, \ldots, \mu_{s(B_\mu)}$, are called *mesh bonds*.

The existence of these nodal and mesh bonds provides the basis for developing pseudo-nodal and pseudo-mesh techniques for the analysis of bondgraph models of physical systems. These bonds may be used to define pseudo-nodal and pseudo-mesh variables, however, further investigation of the algebraic properties of nodal and mesh bondgraphs demonstrates that the entire procedure may be performed purely algebraically.

Let β be a base of a matroid, M, with the property that each fundamental cycle of M with respect to β contains a maximum of two base elements. Such a base is called a *Lagrangian base* of M. Dually, a *Lagrangian co-base* of M is a co-base, β^*, with the property that each fundamental co-cycle of M with respect to β^* contains a maximum of two co-base elements. A Lagrangian base of the cycle matroid of a connected graph, G, corresponds to a Lagrangian tree of G. Similarly, for a non-connected graph a Lagrangian base of the cycle matroid corresponds to a Lagrangian tree for each part of the graph. In the sequel the characteristic algebraic properties of Lagrangian bases and co-bases are exploited, without any consideration of graphs or trees.

Theorem VIII

If B_v is a nodal bondgraph, the set of nodal bonds is a Lagrangian base for the cycle matroid of the extended nodal bondgraph, $M(B'_v)$. Dually, if B_μ is a mesh bondgraph, the set of mesh bonds is a Lagrangian co-base for $M(B'_\mu)$, the cycle matroid of the extended mesh bondgraph.

Proof: By construction there is precisely one external bond on each s-junction of an exended nodal bondgraph and at least one external bond on each p-junction. So colouring red all the external bonds on s-junctions and all the nodal bonds of B'_v provides a base colouring of the extended nodal bondgraph. The corresponding base, β, consists of the set of nodal bonds. Now, with respect to β, the fundamental cycle containing a co-base element which is on a p-junction contains two elements, the co-base bond and the nodal bond on that p-junction. Also, the fundamental cycle of a co-base element which is on an s-junction contains three elements, the co-base bond and the nodal bonds on the two adjacent p-junctions. These comments follow by consideration of the special properties of a nodal bond graph. Hence, the base of nodal bonds is a Lagrangian base. The co-base result follows from duality. ■

Theorem IX

Let B be a graphic bondgraph with equivalent nodal bondgraph B_v. The restriction of the cycle matroid of the extended nodal bondgraph to the non-nodal bonds is a matroid isomorphic to the cycle matroid of B.

Proof: Since the nodal bonds are all incident on p-junctions, restriction of the cycle matroid of B'_v to non-nodal bonds may be accomplished by deleting the nodal bonds from the extended nodal bondgraph. The resulting bondgraph is the nodal bondgraph which, by assumption, is equivalent to B. ■

Let B be a connected graphic bondgraph with equivalent nodal bondgraph B_v. The extended nodal bondgraph, B'_v, can be used to construct an associated graph, G, for the nodal bondgraph. This is done by drawing a Lagrangian tree with edges in one-to-one correspondence to the nodal bonds of B'_v and then connecting one edge for each of the co-base bonds according to the fundamental cycles of B'_v with respect to the nodal base. It is possible to construct the graph in this way because the nodal base is a Lagrangian base of B'_v and this corresponds to a Lagrangian tree of a graph associated with B'_v. The graph G is called the *nodal graph of B with respect to B_v* and is denoted by $G(B_v)$. The edges of the Lagrangian tree are called *nodal edges* and the common vertex of this tree is called a *datum*. If B is not connected then $G(B_v)$ is constructed in parts, one for each part of B, in the manner described above.

Theorem X

Let B be a graphic bondgraph with equivalent nodal bondgraph B_v. Deleting all the nodal edges of the nodal graph of G with respect to B_v leaves a graph which is associated with B.

Proof: Deletion of the nodal edges from the nodal graph corresponds to restriction of the cycle matroid of B'_v to the set of non-nodal bonds. This restricted matroid is the cycle matroid of B, by Theorem IX, and so the restricted nodal graph is associated with B. ■

Example 5. Consider the nodal bondgraph B_v shown in Fig. 6 with its extended nodal bondgraph. The fundamental cycles of B'_v with respect to these nodal bonds

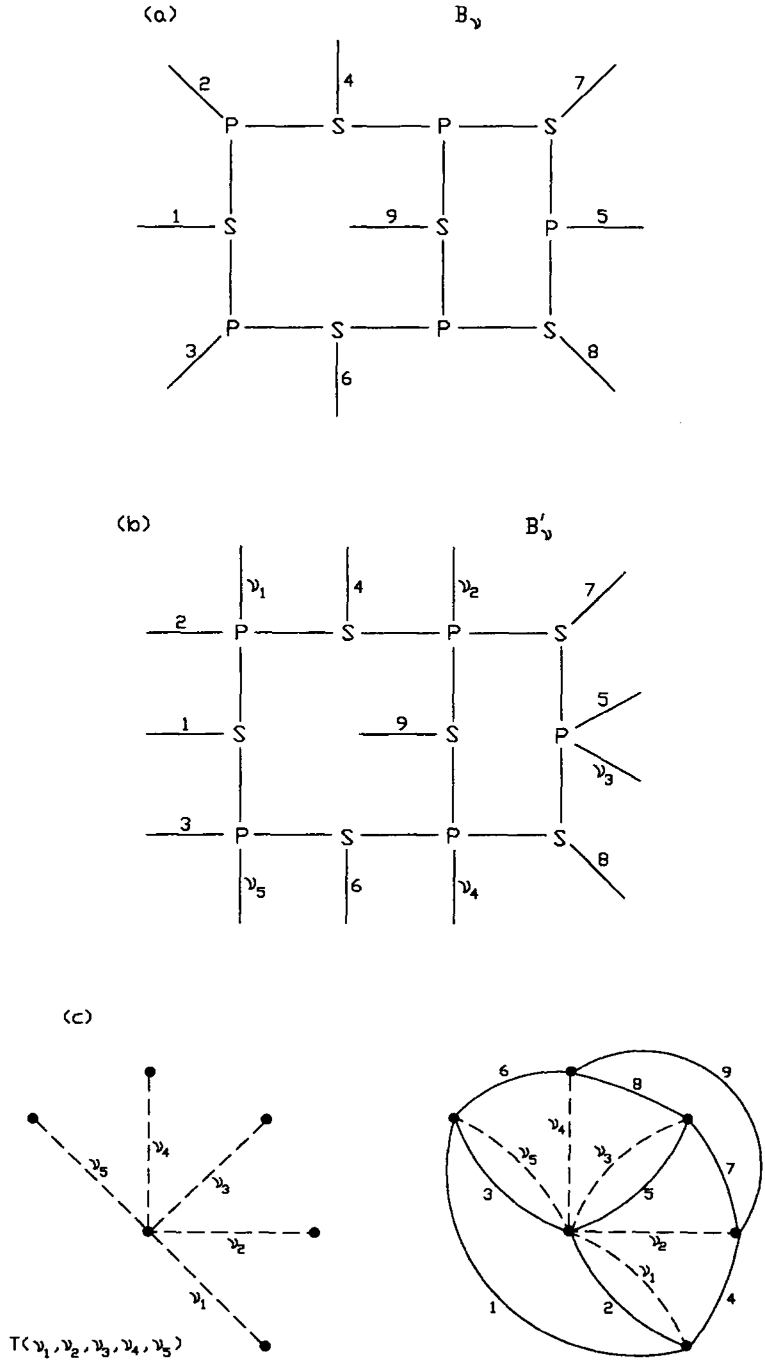


FIG. 6. Nodal bondgraph and nodal graph: (a) nodal bondgraph B_ν; (b) extended nodal bondgraph B'_ν; (c) Lagrangian tree of nodal edges and nodal graph constructed from B_ν.

are read directly from the extended nodal bondgraph:

$$1\nu_1\nu_5, 2\nu_1, 3\nu_5, 4\nu_1\nu_2, 5\nu_3, 6\nu_4\nu_5, 7\nu_2\nu_3, 8\nu_3\nu_4, 9\nu_2\nu_4.$$

The nodal graph of B_ν is constructed in Fig. 6(c) using a Lagrangian tree of nodal edges.

An equivalent mesh bondgraph can be constructed by extracting the hex sub-bondgraph shown in Fig. 7(a). This hex or ring bondgraph may easily be transformed into a number of equivalent bondgraphs using the result of (**4**). By choosing the equivalent hex sub-bondgraph in Fig. 7(b) in which the connecting internal bonds o_1 and o_2 move onto s-junctions, the internal p-junctions may be eliminated, as required for a mesh bondgraph. The equivalent hex sub-bondgraph is re-connected in Fig. 7(c) which, after simplification, gives the mesh bondgraph of Fig. 7(d), B_μ, equivalent to B_ν. By inspection of the extended mesh bondgraph in Fig. 7(e), the fundamental co-cycles with respect to the Lagrangian co-base mesh bonds are:

$$1\mu_1, 2\mu_1\mu_2, 3\mu_1\mu_4, 4\mu_2, 5\mu_2\mu_4, 6\mu_4, 7\mu_2\mu_3, 8\mu_3\mu_4, 9\mu_3.$$

Example 6. Figure 8 shows a bondgraph B and its transformation into an equivalent nodal bondgraph using a hex sub-bondgraph. Inspection of the extended nodal bondgraph gives directly the following fundamental cycles:

$$1\nu_4, 2\nu_2\nu_4, 3\nu_1, 4\nu_3, 5\nu_3\nu_4, 6\nu_1\nu_2, 7\nu_2\nu_3, 8\nu_1.$$

The nodal graph of B with respect to B_ν is shown in Fig. 8(f).

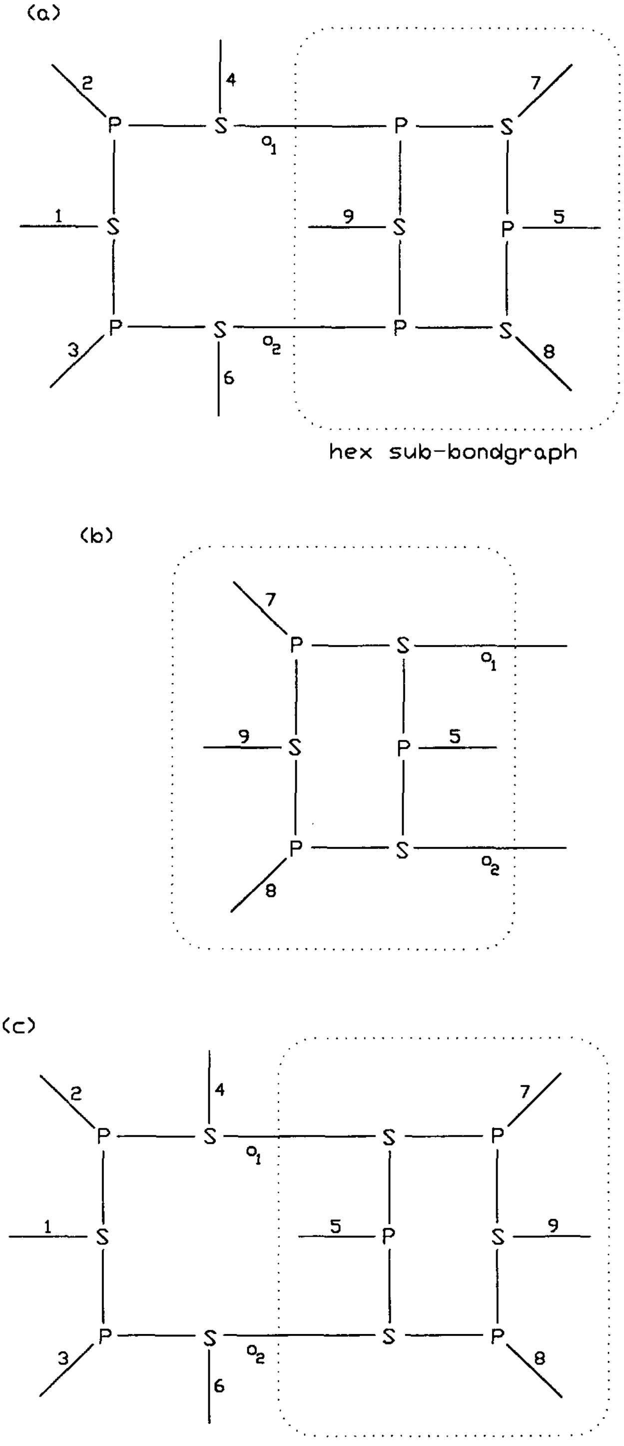
Constructing a mesh bondgraph equivalent to the initial bondgraph B is very simple and requires no manipulation. Inspection of the extended mesh bondgraph B'_μ shown in Fig. 8(g) gives the following fundamental co-cycles with respect to the Lagrangian co-base:

$$1\mu_2\mu_4, 2\mu_2\mu_3, 3\mu_1, 4\mu_4, 5\mu_3\mu_4, 6\mu_2, 7\mu_3, 8\mu_1\mu_2.$$

The graph in Fig. 8(h) is a *mesh graph* for B'_μ in which the Lagrangian co-base corresponds to the four bounded faces of the graph.

IV. Summary and Conclusions

A detailed analysis of the special properties of graphic and co-graphic bondgraphs has been presented. A graphic bondgraph is one which has an associated graph, meaning that they both represent the same collection of cycles and co-cycles. The theory of combinatorial bondgraphs has been used to provide a theoretical basis for the systematic method of obtaining a bondgraph model of a physical system. In this method, the vertices of a graph (or circuit diagram) are replaced by p-junctions and the edges by s-junctions with an extra external edge. The resulting bondgraph is called a vertex bondgraph. The dual vertex bondgraph, as would be used in a systematic mechanical model, is constructed by interchanging the reference to s- and p-junctions. If the graph is a planar graph, its dual graph may also be used to construct a vertex bondgraph, which is then the face bondgraph of the original graph. A systematic bondgraph is a vertex or face bondgraph or either of


(a)
hex sub-bondgraph
(b)
(c)

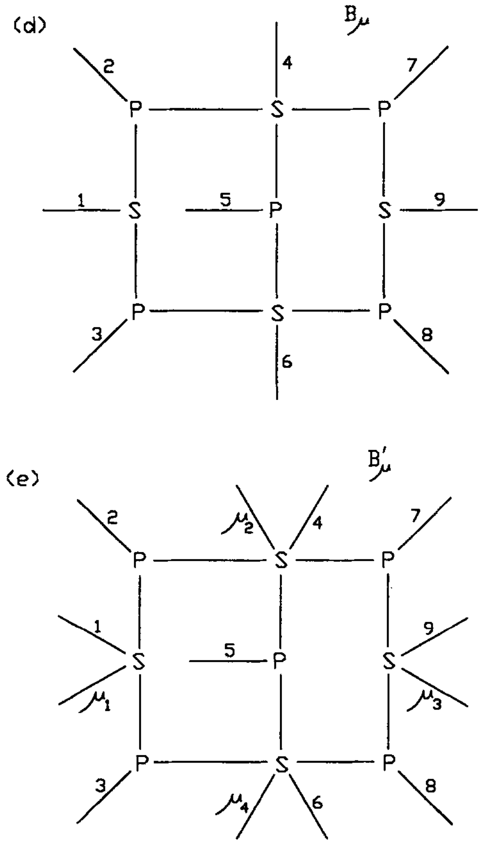


FIG. 7. Construction of a mesh bondgraph from a nodal bondgraph: (a) extraction of hex sub-bondgraph; (b) equivalent hex sub-bondgraph; (c) re-connection of hex sub-bondgraph; (d) simplification to mesh bondgraph B_μ; (e) extended mesh bondgraph B'_μ.

the duals. A vertex or face bondgraph is associated with the original graph, meaning that they represent the same collection of cycles, called the cycle matroid.

The redundant information in a systematic bondgraph may be eliminated by removing one internal junction in each part of the bondgraph, together with all its incident internal bonds. This procedure, called "grounding a node" in the bondgraph literature, does not change the cycle matroid of the bondgraph. In terms of an augmented graph, grounding a vertex corresponds to collapsing the auxilliary vertex onto one of the other vertices of the graph. Various examples were given showing the construction and reduction of vertex and face bondgraphs and illustrating their associated graphs. The reduction of a systematic bondgraph by grounding an auxilliary vertex is a purely combinatorial procedure. When applied to a bondgraph model of a physical system there is no physical interpretation of this procedure and it is incorrect to regard the grounded vertex as a datum for

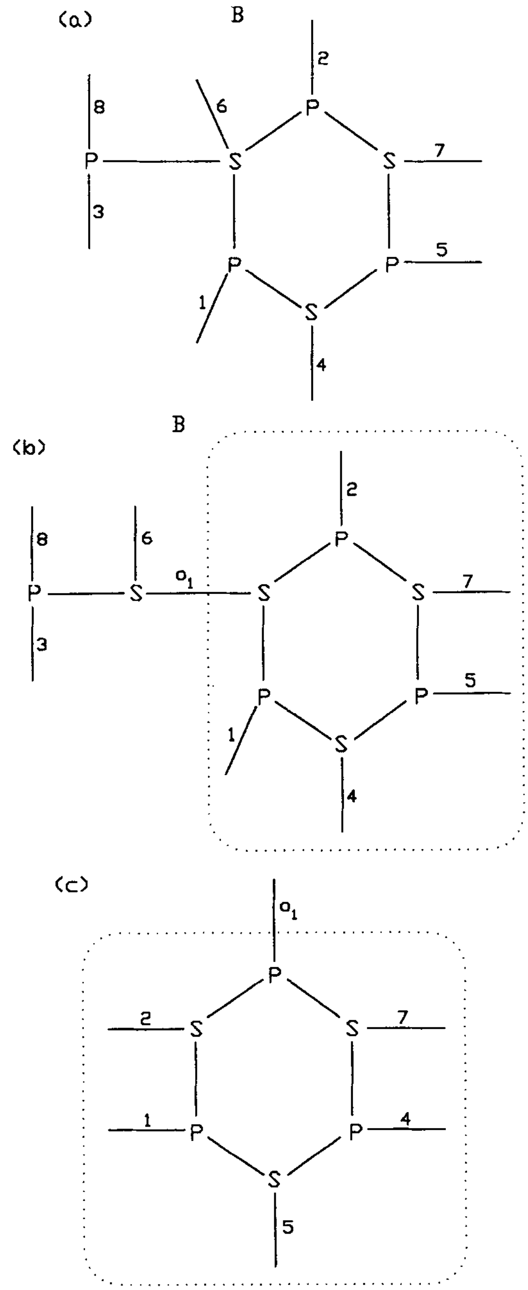


FIG. 8. Bondgraph and construction of equivalent nodal and mesh bondgraphs and graphs: (a) bondgraph B; (b) extracted hex sub-bondgraph; (c) equivalent hex sub-bondgraph with o_1 on a p-junction; (d) sub-bondgraph re-connected to nodal bondgraph B'_v; (e) extended nodal bondgraph B'_v; (f) nodal graph of B with respect to B_v; (g) extended mesh bondgraph B'_μ equivalent to B; (h) mesh graph of B with respect to B_μ. *Continued opposite and overleaf.*

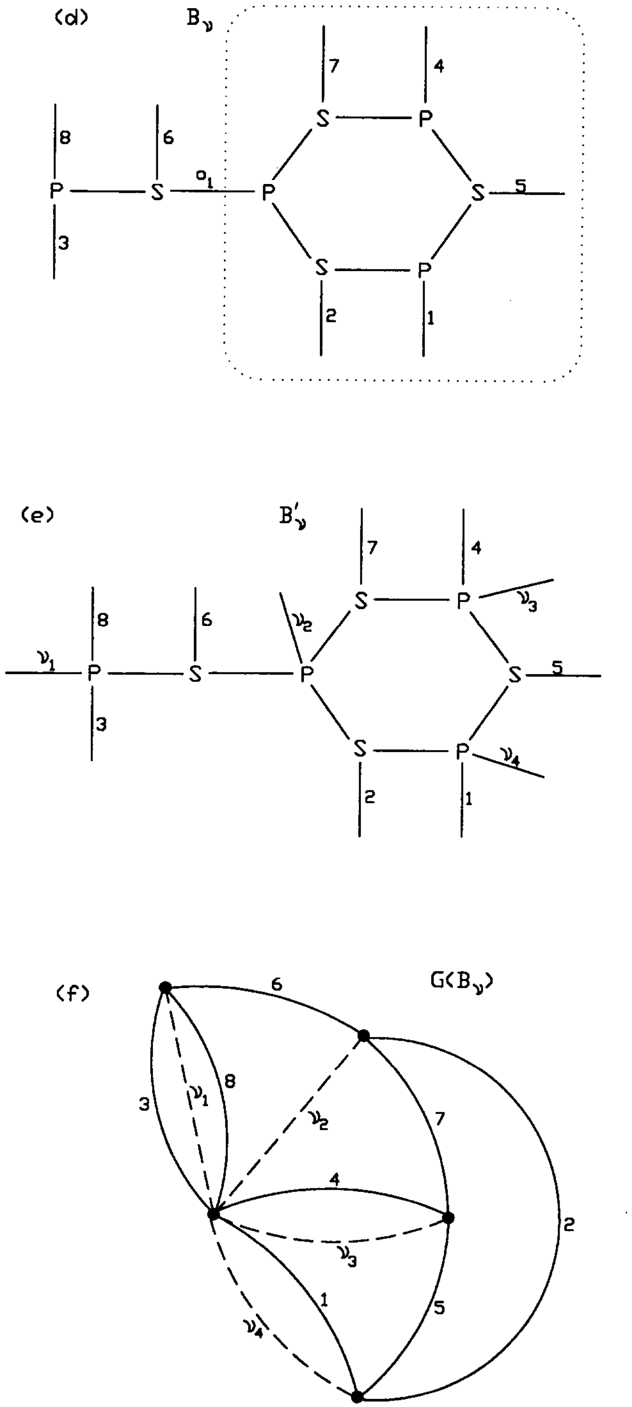
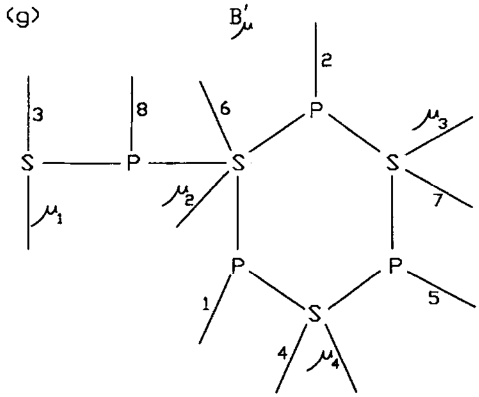


FIG. 8. *Continued.*



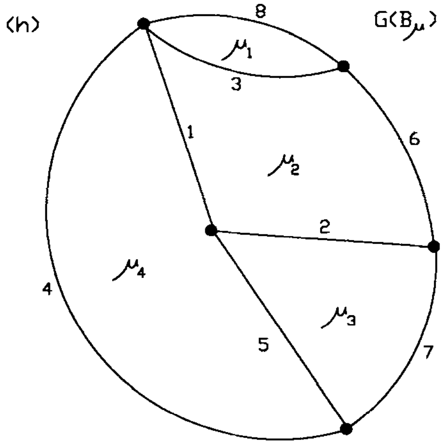


FIG. 8. *Continued.*

physical measurements of the element variables. Only redundant internal combinatorial information is eliminated by reduction.

Special bondgraphs were defined, called nodal and mesh bondgraphs, and their extended versions, which have extra external bonds called nodal and mesh bonds, respectively. Such bondgraphs possess a combinatorial structure which parallels the vertex structure of a graph. It was shown that a bondgraph is graphic (cographic) if and only if it has an equivalent nodal (resp. mesh) bondgraph. Various examples were given showing how equivalent nodal and mesh bondgraphs may be constructed for a given bondgraph. Nodal and mesh bonds provide the basis for developing pseudo-nodal and pseudo-mesh techniques for the analysis of bondgraph models. Although this type of formulation may be carried out on a bondgraph model of a physical system purely algebraically, without the necessity of producing a nodal or mesh bondgraph, their existence establishes that the techniques are available for any graphic or co-graphic bondgraph. This may be important for it provides an alternative formulation to the state space formulation for bondgraph models and the computational advantages of nodal techniques are well-known in network analysis. Also nodal (and mesh) equations have a pre-determined structure which may be automatically assembled directly from the visual diagram (i.e. a bondgraph model). Pseudo-nodal and pseudo-mesh techniques for bondgraph models have been developed as a part of a general combinatorial method of analysing physical systems and a presentation of these results is under preparation.

It should be noted that the relationships between bondgraphs and graphs described herein are included for purposes of analysing the very close connections between these combinatorial diagrams. The bondgraph results stand separately and exist independently of any consideration of graphs. In particular, it has been shown that nodal and mesh techniques are applicable to bondgraphs without the use of graphs or networks and the combinatorial diagrams may be manipulated as required.

References

(1) G. Kron, "Graphs as illegitimate models of electrical networks", *Matrix Tensor Q.*, Vol. 12, pp. 1–10, 1961.
(2) S. H. Birkett and P. H. Roe, "The mathematical foundations of bond graphs I—algebraic theory", *J. Franklin Inst.*, Vol. 326, pp. 329–350, 1989.
(3) S. H. Birkett and P. H. Roe, "The mathematical foundations of bond graphs II—duality", *J. Franklin Inst.*, Vol. 326, pp. 691–708, 1989.
(4) S. H. Birkett and P. H. Roe, "The mathematical foundations of bond graphs III—matroid theory", *J. Franklin Inst.*, Vol. 327, pp. 87–108, 1990.
(5) S. H. Birkett and P. H. Roe, "The mathematical foundations of bond graphs IV—matrix representations and causality", *J. Franklin Inst.*, Vol. 327, pp. 109–128, 1990.
(6) J. U. Thoma, "Simulation by Bondgraphs", Springer, Berlin, 1990.
(7) D. J. Welsh, "Matroid Theory", Academic Press, London, 1976.

Appendix: Summary of the Mathematical Foundations of Bondgraphs

The complete theory of combinatorial bondgraphs is developed in Birkett and Roe (2–5) and the notation and results of these papers are freely used here. As a reference a brief outline is provided in this appendix.

Combinatorial bondgraphs

A *combinatorial bondgraph* consists of junctions of two types, labelled with the letters 'p' and 's', and (undirected) lines incident on either one junction, called an *external bond*, or incident on two junctions, called an *internal bond*. The e external bonds are labelled $1, 2, \ldots, e$ and the i internal bonds are labelled $o_1, o_2, \ldots, o_i$. For a bondgraph B the number of p-junctions is denoted by $p(B)$ or p and the number of s-junctions is $s(B)$ or s. An *internal junction* is one which has no incident external bond. The *dual bondgraph* B^* of B is the bondgraph formed by reversing the labels on all the junctions of B.

Vector spaces of a bondgraph

An algebraic structure is placed on a bondgraph by considering sets of bonds and using the *symmetric difference* of two bond sets, consisting of the elements in one of the sets or the other but not in both. There are two types of *elementary junction vectors*: s-type is all the bonds on a particular s-junction and p-type is any two bonds on a particular p-junction. The duals of these are the *dual elementary junction vectors*: p^*-type is all the bonds on a p-junction and s^*-type is any two bonds on an s-junction. By using symmetric differences the elementary junction vectors generate a vector space W_s called the *s-space* of B (dual of this is the p-space W_p). Restriction of the s-space to external bonds gives a subspace called the *cycle space* of B (dual is the *co-cycle space*). The cycle and co-cycle spaces abstract the basic combinatorial information which a standard bondgraph physical model contains.

Matroids

A *matroid M* on a set S is a collection of subsets of S, called *circuits*, with properties analogous to the properties of cycles or co-cycles. In a combinatorial physical model there is a generalized Kirchhoff law which applies to the cycles and a dual law which applies to the co-cycles. The *cycle matroid* $M(B)$ of a bondgraph B is the matroid whose cicuits are the cycles of B (dual is the *co-cycle matroid*). Two bondgraphs are *equivalent* if they have the same cycle matroid. Several useful transformations of bondgraphs were proved in (**2**), in particular the *diamond equivalence* in which neighbouring s- and p-junctions are *coupled* into a diamond shaped bondgraph with a loop; *ring* or *hex bondgraphs* consisting of six alternating s- and p-junctions arranged in a loop may have the bonds in opposite pairs interchanged to give a selection of equivalent bondgraphs. These operations are useful for manipulating bondgraphs, and examples are given in (**2**).

Connection with graphs

The theory of bondgraphs exists independently of graph theory but there are many close connections between the two theories and these were fully explored in (**2–5**). The *cycle matroid* $M(G)$ of a graph G is defined by the cycles of the graph (sometimes called circuits). A bondgraph B and a graph $G(B)$ are *associated* if they have the same cycle matroid. A graph G may be pictured by a diagram, called a planar representation of G, and whenever bondgraphs associated with a graph are constructed a planar representation is used. An *augmented graph* for a bondgraph B is one which also contains edges corresponding to the internal bonds of B. Eliminating the internal edges of an augmented graph gives a graph associated with B, also called an *external graph* for B. A bondgraph is *graphic* if there exists an associated graph and *co-graphic* if the dual bondgraph has an associated graph.

Causality

An *independent set* of the cycle matroid $M(B)$ is a set of bonds which does not contain a cycle. A *base* of $M(B)$ is a maximal independent set and the number of bonds in any base β is $\rho(B)$, called the *rank* of the bondgraph. The duals of these are a *co-base* β^* and the *co-rank* $\rho^*(B)$. Formulas for the rank and co-rank of a bondgraph are:

$$\rho(B) = p - s + e_s$$

$$\rho^*(B) = s - p + e_p$$

where e_s and e_p are the number of external bonds on s- and p-junctions respectively.

A *pseudo-base colouring* of a bondgraph B is a division of the bonds of B into *red* and *white* bonds so that precisely one red bond is incident on each p-junction and precisely one white bond is incident on each s-junction. This colouring is exactly the same as the assignment of causal strokes in standard bondgraph modelling. A *pseudo-base set* with respect to a particular pseudo-base colouring consists of all the red external bonds on p-junctions and all the white external bonds on s-junctions. A *red–white loop* (causal loop) consists of a loop in which the bonds are alternately coloured red and white. A pseudo-base set β is a base of B if the colouring defining β contains no red–white loop.

Let β be a base of the cycle matroid of B. Its complement is a co-base β^* for B. For each base bond j there is a unique co-cycle of B, called the *fundamental co-cycle* of j with respect to β^*, which contains j and no other base bonds. Dually for each co-base bond k there is a unique cycle, called the *fundamental cycle* of k with respect to β, which contains k and no other base bonds.

Received: 20 August 1992
Accepted: 16 October 1992